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Linear Quadratic Minimax Controllers

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Minimax methods are proposed for the analysis and design of controllers for the best controls with the worst initial conditions, worst parameter changes with specified quadratic norms, and worst disturbances with specified integral-square norms. The worst initial conditions are the only forcing functions; disturbances are regarded as an added set of feedback controls whose magnitudes are limited by negative weights in the performance index. The minimax value of the performance index is easily calculated as the maximum eigenvalue of a steady-state Lyapunov or Riccati matrix. There is a lower bound on the disturbance weights in the performance index; at this bound, the controller design is identical to the H_∞ controller design. There is also an upper bound on the norm of the parameter changes; at this value, the closed-loop system goes unstable, and the corresponding parameter change vector is almost the same as the corresponding vector obtained by "real μ" analysis—the only difference being the use of a quadratic norm instead of an infinity norm.

Introduction

LINEAR quadratic Gaussian (LQG) design methods may produce controllers that are not robust to plant parameter changes, particularly for plants with lightly damped vibration modes, where they design "notch" compensators that are closely tuned to the plant model. Also, for many problems the statistical properties of the process and measurement disturbances are not known well enough to use LQG methods. Thus, there has been a great interest in developing controller design methods that 1) are robust to plant parameter changes (e.g., Doyle's structured singular value of μ synthesis[1] and 2) depend only on some amplitude characterization of the disturbances (H_∞ methods[2]). However, these latter methods are relatively complicated to use, and there is still no efficient algorithm for real μ analysis or synthesis.

In this paper, we develop time-domain methods to analyze controlled response to worst plant parameter changes having a specified quadratic norm and worst disturbances having a specified integral-square norm. These methods use only standard control analysis algorithms available in current professional software for solving Lyapunov and Riccati equations.

The only forcing function is the initial condition vector whose magnitude is fixed at unity, but whose "direction" maximizes a quadratic performance index. Similarly, the magnitude of the deviation of the parameter vector from nominal is specified, but the direction of this vector maximizes the performance index. Disturbances are regarded as unfriendly bounded controls that maximize the performance index, whereas the bounded control vector minimizes the performance index.

New Performance Criterion J_w

For a stable linear dynamic system,

$$\dot{x} = Ax, \qquad x(0) = x_0 \tag{1}$$

$$y = Cx \tag{2}$$

J_w is defined as

$$\max_{x_0} \frac{\int_0^\infty y^T Q y \, \mathrm{d}t}{x_0^T x_0} \tag{3}$$

From simple dynamic programming, J_w is equal to

$$\max_{x_0} \frac{x_0^T S x_0}{x_0^T x_0} \tag{4}$$

where S is the solution of the steady-state Lyapunov equation

$$SA + A^T S + C^T Q C = 0 \tag{5}$$

The obvious solution of Eq. (4) is

$$J_w = \max[\mathrm{eig}(S)] \tag{6}$$

and the worst initial condition x_0 is the corresponding unit eigenvector of S:

$$x_0 = \mathrm{eigvec}(S) \tag{7}$$

J_w measures performance to the worst possible direction of the unit initial condition vector. It is the H_∞ norm of S and tends to infinity as A tends toward instability.

System with a Given State Feedback Controller

Consider a system with state feedback:

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0 \tag{8}$$

$$y = Cx \tag{9}$$

$$u = -Kx \tag{10}$$

We wish to find

$$J_w = \max_{x_0} \frac{\int_0^\infty (y^T Q y + u^T R u) \, \mathrm{d}t}{x_0^T x_0} \tag{11}$$

From the previous section J_w is given by determining S from the Lyapunov equation:

$$S(A - BK) + (A - BK)^T S + C^T Q C + K^T R K = 0 \tag{12}$$

and J_w and x_0 are given by Eqs. (6) and (7).

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Minimax State Feedback Design

The optimal state feedback gain matrix can be found by minimizing J_w with respect to K. For a given x_0, minimization of the quadratic cost in Eq. (3) using Eq. (12) yields

$$K = R^{-1}B^TS \tag{13}$$

which converts the Lyapunov equation [Eq. (12)] into the LQR Riccati equation:

$$SA + A^TS + C^TQC - SBR^{-1}B^TS = 0 \tag{14}$$

Since K is independent of x_0, it also minimizes J_w; i.e., the minimax state feedback design coincides with the LQR design.

System with a Given Compensator

Consider the system (8) and (9) with a dynamic compensator:

$$\dot{x}_c = A_cx_c + B_cy_s, \qquad x_c(0) = 0 \tag{15}$$

$$u = C_cx_c \tag{16}$$

where

$$y_s = C_sx \tag{17}$$

is the measurement vector. Again, y and u are proportional to x_0; thus, Eq. (11) is still an appropriate performance criterion.

To find J_w we augment the dynamic system with the compensator equations; i.e., we replace x by x_a, where

$$x_a = \begin{bmatrix} x \\ x_c \end{bmatrix} \tag{18}$$

It follows that J_w and x_0 are again given by Eqs. (6) and (7), where now S is determined from S_a as follows:

$$S_aA_a + A_a^TS_a + Q_a = 0 \tag{19}$$

where

$$A_a = \begin{bmatrix} A & BC_c \\ B_cC_s & A_c \end{bmatrix} \tag{20}$$

$$Q_a = \begin{bmatrix} C^TQC & 0 \\ 0 & C_c^TRC_c \end{bmatrix} \tag{21}$$

$$S_a = \begin{bmatrix} S & (\cdot) \\ (\cdot) & S_c \end{bmatrix} \tag{22}$$

Minimax Compensator Design

Since there is no measurement or process noise in this simplified formulation and the parameters are (implicitly) assumed to be known perfectly, the optimal estimator is a differentiator; i.e., the available measurements are differentiated as many times as necessary to reconstruct the states using the plant model. These estimated states are then fed back with the same gains K_r that were developed for the state feedback case.

Such a differentiator can be realized approximately by using fast observers, i.e., observers with eigenvalues that are fast compared to the controller bandwidth. This is the result achieved when loop-transfer recovery[3] is used. Since real systems have noise and the parameters are not known perfectly, this method is unlikely to give good results in practice.

Max J_w with a Norm on Disturbances

Consider a system with a disturbance input $w(t)$:

$$\dot{x} = Ax + \Gamma w, \qquad x(0) = x_0 \tag{23}$$

$$y = Cx \tag{24}$$

We wish to find $w(t)$ and x_0 to maximize

$$J = \int_0^\infty y^TQy \, dt \tag{25}$$

subject to the following constraints:

$$W = \int_0^\infty w^TR_ww \, dt \tag{26}$$

$$1 = x_0^Tx_0 \tag{27}$$

We adjoin the constraints to the performance criterion with Lagrange multipliers μ and ν:

$$\bar{J}_w = \max_{w(t),x_0} \left\{ \int_0^\infty (y^TQy - \mu w^TR_ww) \, dt + \mu W - \nu(x_0^Tx_0 - 1) \right\} \tag{28}$$

This produces an integral quadratic penalty on $w(t)$ with a negative scalar weight, $-\mu$, where $\mu>0$; thus, the second derivative of the integrand with respect to w is negative, as it must be for maximization. The choice of μ determines W in Eq. (26) and vice versa, and R_w is chosen to select the relative levels of the elements of w.

The w may be interpreted as a "control" that is trying to destabilize the system rather than stabilize it. This problem is identical in form to the optimal state feedback problem discussed earlier. Thus, the solution for x_0 is again Eq. (7), with S determined by the Riccati equation:

$$SA + A^TS + C^TQC + (1/\mu)S\Gamma R_w^{-1}\Gamma^TS = 0 \tag{29}$$

and w is given by

$$w = K_w \cdot x \tag{30}$$

where

$$K_w = (1/\mu)R_w^{-1}\Gamma^TS \tag{31}$$

However, J_w is given by

$$J_w = \max\left[\mathrm{eig}(S)\right] + \mu W \tag{32}$$

where μ is determined so that

$$W = \mathrm{tr}\left[P \cdot (K_w^TR_wK_w)\right] \tag{33}$$

and P is the solution to the Lyapunov equation:

$$0 = (A + \Gamma K_w)P + P(A + \Gamma K_w)^T + x_0x_0^T \tag{34}$$

As μ is varied, W varies. Note that

$$w \to 0 \quad \text{as} \quad \mu \to \infty \tag{35}$$

System with a Given State Feedback Controller

In this case Eqs. (23–25) are replaced by

$$\dot{x} = Ax + Bu + \Gamma w \tag{36}$$

$$u = -Kx \tag{37}$$

$$J = \int_0^\infty (y^TQy + u^TRu) \, dt \tag{38}$$

The obvious modification to Eq. (29) is to replace A with $A - BK$ and C^TQC with $C^TQC + K^TRK$:

$$S(A - BK) + (A - BK)^TS + C^TQC + K^TRK + 1/\mu \, S\Gamma R_w^{-1}\Gamma^TS = 0 \tag{39}$$

Minimax State Feedback Design

The optimal state feedback gain matrix K can be found by minimizing J with respect to u while simultaneously maximizing it with respect to w and x_0. It is straightforward[4,5] to show that the optimal value of K is again Eq. (13), so that the Riccati equation (39) is modified to

$$SA + A^TS + C^TQC - S\left(BR^{-1}B^T - \frac{1}{\mu}\Gamma R_w^{-1}\Gamma^T\right)S = 0 \quad (40)$$

As $\mu \to \infty$ (disturbances become "expensive"), then $w \to 0$, and Eq. (40) becomes the LQR Riccati equation.

System with a Given Compensator

Consider the system (36) with the measurement vector

$$y_s = C_s x + v \quad (41)$$

and the dynamic compensator [Eqs. (15) and (16)].

The $v(t)$ is another disturbance vector (an additive sensor error) whose magnitude we constrain by adding another term in Eq. (28); the worst disturbances, along with the worst initial conditions, for a given compensator are found by considering

$$\bar{J}_w = \max_{x_0, w, v} \int_0^\infty (y^TQy + u^TRu) - \mu(w^TR_ww + v^TR_vv)\,\mathrm{d}t + \mu W - \nu(x_0^Tx_0 - 1) \quad (42)$$

Note that the worst disturbances are feedbacks on both x and x_c, whereas the controls are feedbacks only on x_c.

If we consider the augmented state vector x_a, this problem is identical in form to an LQR problem where the controls are the disturbances $w(t)$ and $v(t)$, which *destabilize* rather than *stabilize* the system.

The worst disturbances are feedbacks of the augmented state x_a, where the gain matrices are determined by the solution of the Riccati equation:

$$S_aA_a + A_a^TS_a + Q_a + (1/\mu)S_a\Gamma_aR_a^{-1}\Gamma_a^TS_a = 0 \quad (43)$$

where A_a, Q_a, and S_a were defined in Eqs. (20–22), and

$$\Gamma_a = \begin{bmatrix} \Gamma & 0 \\ 0 & B_c \end{bmatrix} \quad (44)$$

$$R_a = \begin{bmatrix} R_w & 0 \\ 0 & R_v \end{bmatrix} \quad (45)$$

$$w = K_w \cdot x_a \quad (46)$$

$$v = K_v \cdot x_a \quad (47)$$

where

$$\begin{bmatrix} K_w \\ K_v \end{bmatrix} = \begin{bmatrix} K_{wx} & K_{wc} \\ K_{vx} & K_{vc} \end{bmatrix} = \frac{1}{\mu}\begin{bmatrix} R_w^{-1}\Gamma^T & 0 \\ 0 & R_v^{-1}B_c^T \end{bmatrix} S_a \quad (48)$$

This implies that

$$y_s = [C_s + K_{vx}]x + K_{vc}x_c \quad (49)$$

Thus, the disturbance $v(t)$ provides continuously changing scale factors and biases to the measurements.

The worst x_0 is again given by Eq. (7), where S is a partition of S_a; this occurs because we assumed $x_c(0) = 0$.

However, J_w is given by

$$J_w = \max\left[\mathrm{eig}(S)\right] + \mu W \quad (50)$$

where W may be determined from

$$W = \mathrm{tr}\left[P_a \cdot (K_a^TR_aK_a)\right] \quad (51)$$

and P_a is the solution to the Lyapunov equation:

$$0 = (A_a + \Gamma_aK_a)P_a + P_a(A_a + \Gamma_aK_a)^T + x_{a0}x_{a0}^T \quad (52)$$

and

$$K_a = \begin{bmatrix} K_w \\ K_v \end{bmatrix} \quad (53)$$

Minimax Compensator Design

This is an area for future research. We expect this solution (if it exists) to be similar to the compensator for a given μ found in Refs. 2 and 6; however, initial conditions were specified in those references. The relationship of minimax compensator design to H_∞ design is discussed in Appendix B.

Max J_w with a Norm on Parameter Changes

Let p be a vector of plant parameters, where

$$A = A(p), \qquad B = B(p), \qquad C = C(p) \quad (54)$$

and let p_{nom} be the nominal value of p. Let us consider variations of the parameters

$$p = p_{\mathrm{nom}} + \Delta p \quad (55)$$

where a scalar measure of simultaneous changes in all parameters is

$$\sigma(\Delta p) = \left\{(\Delta p)^T\Sigma^{-2}\Delta p\right\}^{1/2} \quad (56)$$

With a Gaussian interpretation of the parameter changes,

$$\Sigma = \text{diagonal matrix of standard deviations} \quad (57)$$

An interesting parameter robustness problem is to determine Δp to maximize J_w with a specified value of σ in Eq. (56), i.e., to find the worst J_w with a given norm σ on the parameter changes Δp.

Appendix A shows that a necessary condition for the maximum is

$$\Delta p \equiv p - p_{\mathrm{nom}} = \sigma \cdot \Sigma \cdot \alpha \quad (58)$$

where α is a unit vector in the direction of the gradient:

$$J_p = \Sigma \cdot \left(\frac{\partial J_w}{\partial p}\right)^T \quad (59)$$

which is evaluted at p. To satisfy this necessary condition, we used the following iterative algorithm:

1) Specify σ.
2) Put $\Delta p = 0$ and $\alpha = 0$.
(*) Compute α_{new} using Eq. (59).
3) If $|\alpha_{\mathrm{new}} - \alpha_{\mathrm{old}}| < \epsilon$, then stop.
4) Compute Δp using (58) and go to (*).

In the examples we have computed, only a few iterations on Δp were required, and α varied slowly over the full range of σ from 0 to σ_∞.

This algorithm may not yield the global minimum. However, it appears to do so in the following example, since the results were checked using Monte Carlo search (Figs. 8 and 9).

To compute J_p in Eq. (59), Appendix A shows that

$$\frac{\partial J_w}{\partial p_i} = 2 \cdot \mathrm{tr}\left[P\left(C^TQ\frac{\partial C}{\partial p_i} + S\frac{\partial A}{\partial p_i}\right)\right] \quad (60)$$

where P is determined by the Lyapunov equation:

$$AP + PA^T + x_0x_0^T = 0 \quad (61)$$

Note

$$P = \int_0^\infty xx^T \, \mathrm{d}t \tag{62}$$

starting with the worst initial condition vector x_0.

As σ is increased, J_w increases; however, there is usually a maximum value of σ where $J_w \to \infty$ and the corresponding value of σ is the stability parameter margin σ_∞ (Ref. 7). Appendix C discusses the relationship of this method of finding a parameter margin with Doyle's real μ analysis.

With a Given State Feedback Controller

For the state feedback case it is straightforward to show that the derivative of J_w with respect to a parameter p_i in A, B, and/or C is given by

$$\frac{\partial J_w}{\partial p_i} = 2 \cdot \mathrm{tr}\left[P\left(C^T Q \frac{\partial C}{\partial p_i} + S\frac{\partial A_{\mathrm{cl}}}{\partial p_i}\right)\right] \tag{63}$$

where

$$0 = SA_{\mathrm{cl}} + A_{\mathrm{cl}}^T S + C^T QC \tag{64}$$

$$0 = A_{\mathrm{cl}}P + PA_{\mathrm{cl}}^T + x_0 x_0^T \tag{65}$$

$$A_{\mathrm{cl}} = A - B \cdot K \tag{66}$$

and J_w and x_0 are again given by Eqs. (6) and (7). P is given by Eq. (62) and is identical to the steady-state covariance matrix of the augmented system if it is forced by white noise with density $x_0 x_0^T$ (Refs. 8 and 9).

Minimax State Feedback Design

If we could minimize J_w with respect to the state feedback gain matrix K while simultaneously maximizing J_w with respect to Δp with a specified norm, we would have the best state feedback controller for the worst parameter change with the specified norm. This would optimize the controller for parameter robustness. This problem is not yet solved.

System with a Given Compensator

For the case with a dynamic compensator the derivative of J_w with respect to a parameter p_i in A_a is given by

$$\frac{\partial J_w}{\partial p_i} = 2 \cdot \mathrm{tr}\left[P_a S_a \frac{\partial A_a}{\partial p_i}\right] \tag{67}$$

where A_a, Q_a, and S_a are given by Eqs. (20–22), and P_a is computed from

$$0 = A_a P_a + P_a A_a^T + x_{a0} x_{a0}^T, \qquad x_{a0} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \tag{68}$$

with x_0 being the worst-case initial plant state conditions.

Minimax Compensator Design

If we could minimize J_w with respect to the compensator parameters K_c while simultaneously maximizing J_w with respect to Δp with a specified norm, we would have the best compensator with the specified order for the worst parameter change with the specified norm. This would optimize the compensator for parameter robustness. This problem is not yet solved. With a norm on disturbances, it is a minimax problem; however, there is no closed-form expression for $J_w(K_c)$ as there is in Eq. (6).

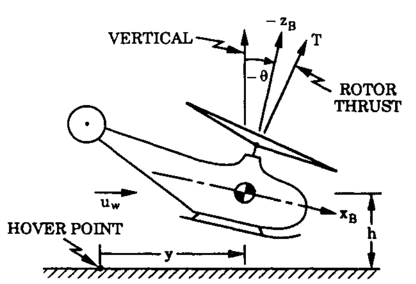

Fig. 1 Nomenclature for helicopter near hover.

Example: Control of a Helicopter in Hover

We consider a helicopter near hover disturbed by horizontal wind gusts (see Fig. 1).

We shall evaluate two types of position-hold autopilot designs: one with full state feedback and the other with a dynamic compensator that uses only a measurement of position. We expect that the former designs will be quite robust and the latter designs quite nonrobust to plant parameter changes.

The plant model is

$$\begin{bmatrix} \dot u \\ \dot q \\ \dot\theta \\ \dot y \end{bmatrix} = \begin{bmatrix} X_u & X_q & -g & 0 \\ M_u & M_q & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ q \\ \theta \\ y \end{bmatrix} + \begin{bmatrix} X_\delta \\ M_\delta \\ 0 \\ 0 \end{bmatrix} \delta + \begin{bmatrix} -X_u \\ -M_u \\ 0 \\ 0 \end{bmatrix} u_w \tag{69}$$

where g is the force per unit mass due to gravity, u the forward velocity, q the pitch angular velocity, θ the pitch angle, y the postion deviation from desired hover point, δ the longitudinal cyclic stick deflection, and u_w the horizontal wind velocity.

We shall take as our performance index

$$J = \int_0^\infty (y^2 + \delta^2) \, \mathrm{d}t \tag{70}$$

where y is in feet and δ is in deci-inches.

The nominal values of the six parameters were taken (for an OH6A helicopter[10]) to be

$$[X_u, X_q, M_u, M_q, X_\delta, M_\delta] = [-0.0257, 0.013, 1.26, -1.765, 0.086, -7.408] \tag{71}$$

where the units are feet, seconds, and centiradians, and δ is in deci-inches.

With a Given State Feedback Controller

The minimax state feedback controller (which is the same as the LQR controller) has state feedback gains

$$K_r = [1.9890, \ -0.2560, \ -0.7589, \ 1.0000] \tag{72}$$

and the worst unit initial condition vector is

$$x_0 = [0.7929, \ -0.0466, \ -0.1871, \ 0.5780] \tag{73}$$

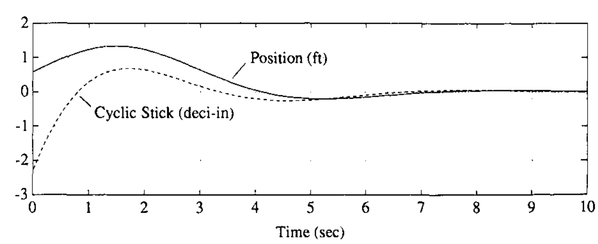

Fig. 2 OH6A helicopter with LQR state feedback controller with worst unit initial condition vector.

The corresponding value of J_w is

$$J_w = 5.581 \tag{74}$$

The x_0 corresponds to a positive position deviation and a positive velocity (i.e., going farther away in the same direction as the position deviation), a nose-down pitch angle, and a pitch rate causing the nose to pitch down some more.

Figure 2 shows the time histories of y and δ for the worst unit initial condition vector x_0.

With a Given Compensator

An LQG compensator was designed assuming a white noise horizontal wind u_w with density = 18 ft^2/s and a measurement of position deviation with an additive white noise error with a density of 0.4 ft^2/s. This yielded the estimator gain vector:

$$K_e = [2.3172, 0.7218, -3.6610, 2.1528]^T \tag{75}$$

The LQG dynamic compensator feeds back the estimated state:

$$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - C\hat{x}) \tag{76}$$

$$u = -K_r\hat{x} \tag{77}$$

The corresponding J_w is

$$J_w = 13.45 \tag{78}$$

and the worst initial condition vector of unit length is

$$x_0 = [0.9010, 0.0409, -0.0915, -0.4222] \tag{79}$$

The x_0 corresponds to a negative position deviation, a positive velocity, and almost negligible pitch angle and pitch rate. This confuses the estimator initially and produces a big rearward overshoot as shown in Fig. 3, which results in a much higher J_w that we obtained for the minimax state feedback (LQR) controller.

With a Given State Feedback Controller and Worst Wind Disturbance with a Specified Norm

We selected $\max|u_w(t)| = 5$ ft/s and found μ by interpolation (with $R_w = 1$). We found $\mu = 0.12$ with the LQR controller using the Riccati equation (39).

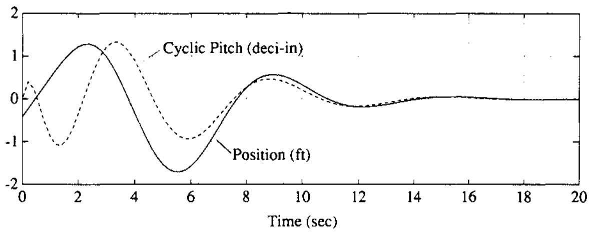

Fig. 3 OH6A helicopter with an LQG compensator using a measurement of position with worst unit initial condition vector.

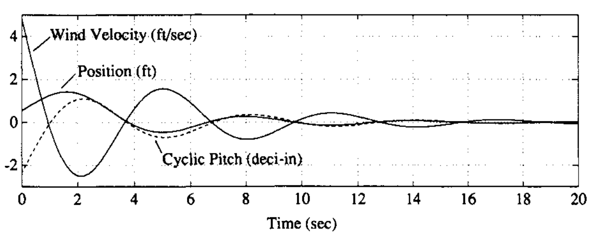

Fig. 4 OH6A helicopter with LQR state feedback controller with worst horizontal wind disturbance and worst unit initial condition vector.

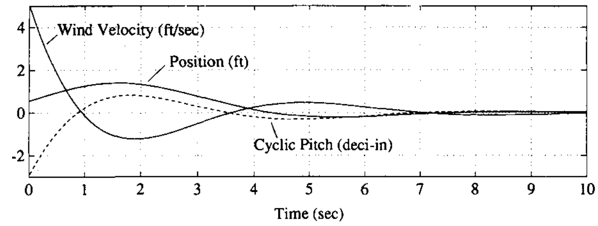

Fig. 5 OH6A helicopter with minimax state feedback controller with worst horizontal wind disturbance and worst unit initial condition vector.

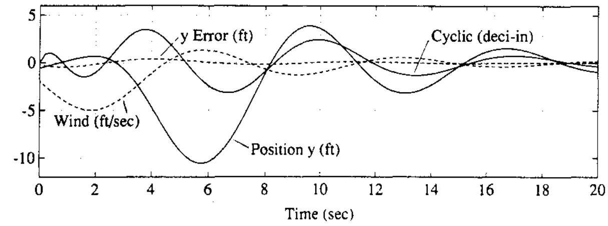

Fig. 6 OH6A helicopter with LQG compensator with worst horizontal wind disturbance $u_w(t)$, worst sensor error $v(t)$, and worst unit initial condition vector.

Figure 4 shows the position deviation y, the cyclic stick δ, and the worst wind history u_w vs time for the LQR controller, with the worst unit initial conditions. The value of J_w increased to 9.06 (from 5.58 with no wind), and the worst unit initial condition vector was

$$x_0 = [0.8108, -0.0529, -0.2040, 0.5460] \tag{80}$$

The worst $u_w(t)$ is given by the positive feedback

$$u_w = K_w \cdot x \tag{81}$$

where

$$K_w = [4.1235, -0.5638, -1.6420, 2.0109] \tag{82}$$

With Minimax State Feedback and Worst Wind Disturbance with a Specified Norm

We again selected $\max|u_w(t)| = 5$ ft/s and found μ by interpolation (with $R_w = 1$); we found $\mu = 0.11$ with the minimax state feedback controller using the Riccati equation (40).

Figure 5 shows position deviation y, cyclic stick δ, and the worst wind history $u_w(t)$ vs time for the minimax state feedback controller, with the worst unit initial conditions. The value of J_w decreased to 7.27 (from 9.06 with the LQR controller), and the worst unit initial condition vector was

$$x_0 = [0.8048, -0.0533, -0.2030, 0.5552] \tag{83}$$

The best control $\delta(t)$ is given by

$$\delta = -K_u \cdot x \tag{84}$$

where

$$K_u = [2.5161, -0.3471, -1.0067, 1.2316] \tag{85}$$

The worst $u_w(t)$ is given by the positive feedback:

$$u_w = K_w \cdot x \tag{86}$$

where

$$K_w = [4.3153, -0.5659, -1.6672, 2.1677] \tag{87}$$

The K_u gains are higher than the LQR gains K_r, and the response is faster (settling time of ~8 s instead of ~25 s). The K_w gains are about the same.

With a Given Compensator and Worst Wind and Sensor Disturbances with Specified Norms

Again we selected $\max|u_w(t)| = 5$ ft/s and found μ by interpolation (with $R_w = 1$, $R_v = 20$); we found $\mu = 3.97$.

Figure 6 shows position deviation y, cyclic stick δ, the worst wind history $u_w(t)$, and the worst position sensor error $v(t)$ vs time for the LQG controller, with the worst unit initial conditions. The value of J_w increased to 364.7 (from 13.5 with no wind or sensor error), and the worst unit initial condition vector was

$$x_0 = [0.8087, 0.1355, 0.0765, -0.5673] \tag{88}$$

The worst $u_w(t)$ and $v(t)$ are given by the positive feedbacks:

$$u_w = K_w \cdot x_a \tag{89}$$

$$v = K_v \cdot x_a \tag{90}$$

where

$$K_w = [-1.9046, -0.6563, -0.8800, 0.5451, 1.9932, 0.6688, 0.8352, -0.4955] \tag{91}$$

$$K_v = [-0.2716, -0.0435, -0.0484, -0.0138, 0.2725, 0.0429, 0.0473, 0.0136] \tag{92}$$

The gains on x are almost the negative of those on $\hat{x}$, so that w and v are nearly proportional to the estimate-error $\hat{x} - x$.

With Worst Parameter Changes with Specified Norms

We take as the standard deviation matrix of the six parameters in Eq. (71):

$$\Sigma = \mathrm{diag}[0.02, 0.01, 0.2, 0.2, 0.02, 0.5] \tag{93}$$

For the LQR state feedback controller the gradient of J_w with respect to these six parameters was calculated as

$$\frac{\partial J_w}{\partial p} \cdot \Sigma = [0.11, 0.19, -0.03, -0.13, -0.10, 0.22] \tag{94}$$

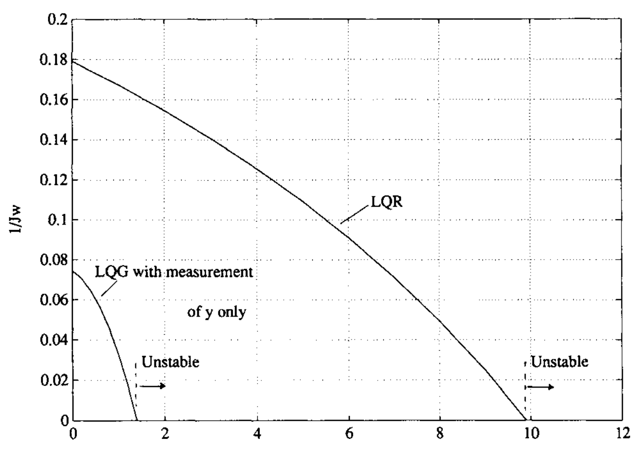

Fig. 7 OH6A helicopter with LQR controller and LQG compensator: $1/J_w$ vs σ for worst parameter changes.

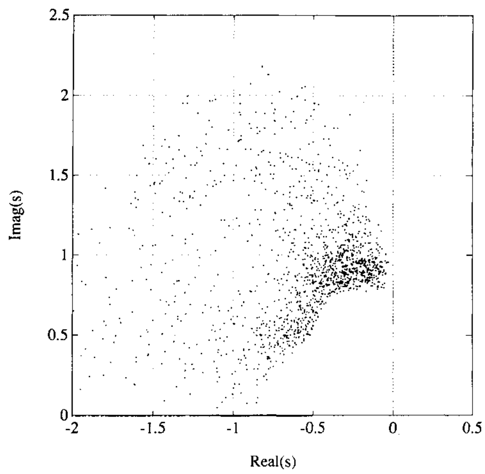

Fig. 8 OH6A helicopter with LQR controller: random locus of roots on the hyperellipsoid $\sigma = 9.9$.

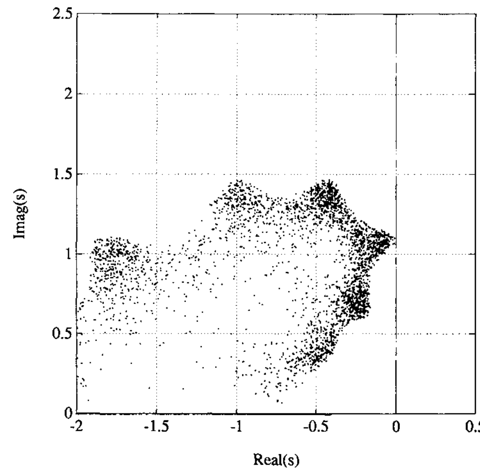

Fig. 9 OH6A helicopter with LQG controller: random locus of roots on the hyperellipsoid $\sigma = 1.47$.

The magnitude of this gradient divided by nominal J_w is 0.063, indicating only a 6.3% increase in J_w for a one-sigma change in the worst direction in the parameter space.

For the LQG controller the gradient of J_w with respect to these six parameters was calculated as

$$\frac{\partial J_w}{\partial p} \cdot \Sigma = [1.51, 1.65, 1.15, 3.60, -0.88, -0.66] \tag{95}$$

The magnitude of this gradient divided by nominal J_w is 0.34, indicating a 34% increase in J_w for a one-sigma change in the worst direction in the parameter space. Thus, the LQG controller starts with worse performance than the LQR controller and degrades much faster with parameter changes.

Figure 7 shows minimum $1/J_w$ vs σ for $p = p_{\mathrm{nom}} + \sigma \cdot \Sigma \cdot \alpha^T$, where α is a unit vector parallel to $\Sigma \cdot \partial J/\partial p$; i.e., α is in the direction of worst parameter change. Note the system with the LQG controller goes unstable for $\sigma = 1.4$, whereas the system with the LQR controller does not go unstable until $\sigma = 9.9$.

Figure 8 shows a locus of roots for the system with the LQR controller on the hyperellipsoid $\sigma = 9.9$, which is the stability parameter margin. Note the "cloud" of points, which are all stable except at $s = 0.90j$, where

$$\Delta p^T \cdot \Sigma^{-1} = [3.2, 4.2, -0.4, 2.4, -4.2, 6.9] \tag{96}$$

This corresponds to increases in M_δ, X_q, X_u, and M_q and a decrease in X_δ. The performance is insensitive to change in M_u.

Figure 9 shows a locus of roots for the system with the LQG controller on the hyperellipsoid $\sigma = 1.47$, which is the stability parameter margin. Again there is a cloud of points, which are all stable except at $s = 1.095j$, where

$$\Delta p^T \cdot \Sigma^{-1} = [0.16, 0.31, 0.34, 1.31, -0.17, -0.43] \quad (97)$$

Thus, it is mainly a 1.31σ increase in M_q that produces instability. The estimator has the wrong parameters (p_{nom}) and gives poor estimates. The performance of the closed-loop system rapidly degrades as the plant parameters change in the worst direction.

Appendix D gives a more direct method of determining the stability parameter margin σ_∞ using the closed-loop characteristic equation. It finds Δp to give the smallest σ for which there is a pole on the $j\omega$ axis. The method is not foolproof [it may not find the global maximum; $\sigma_\infty(\omega)$ might be discontinuous], but we found it useful for checking Figs. 8 and 9.

Conclusions

A new performance criterion for worst case analysis and design was proposed: the value of the usual integral quadratic performance index for the worst initial condition vector with a specified quadratic norm. It is simple to compute and is useful for 1) comparing the performance of different controllers for a given plant, 2) determining the worst disturbance histories with a given integral quadratic norm, and 3) determining the worst parameter changes with a specified quadratic norm. An example was given of a helicopter in hover with several different controllers, worst disturbances, and worst parameter changes.

Appendix A: Derivation of Max J_w with a Norm on Parameter Changes

The problem is to find Δp and x_0 to maximize

$$J = \frac{1}{2}\int_0^\infty y^T Q y \, dt$$

subject to the following constraints:

$$\dot{x} = A(p)x, \qquad x(0) = x_0$$

$$y = C(p)x, \qquad \sigma^2 = \Delta p^T \Sigma^{-2} \Delta p$$

$$\Delta p = p - p_{\text{nom}}, \qquad x_0^T x_0 = 1$$

where p is a vector of parameters.

The solution is obtained by adjoining the constraints to the performance index with Lagrange multipliers ν and μ as follows:

$$\bar{J} = \int_0^\infty \left[\frac{x^T C^T Q C x}{2} + \lambda^T(Ax - \dot{x})\right] dt - \frac{\nu}{2}(\Delta p^T \Sigma^{-2} \Delta p - \sigma^2) - \frac{\mu}{2}(x_0^T x_0 - 1)$$

Consider the differential change of $\bar{J}$ due to differential changes of Δp and x_0 and variations of x:

$$\begin{aligned} d\bar{J} &= \int_0^\infty \left[(x^T C^T Q C + \lambda^T A)\delta x - \lambda^T \delta\dot{x} + \left(x^T C^T Q \frac{\partial C}{\partial p} + \lambda^T \frac{\partial A}{\partial p}\right) d(\Delta p) x\right] dt \\ &\quad - \nu \Delta p^T \Sigma^{-2} \, d(\Delta p) - \mu x_0^T \, dx_0 \\ &= -\lambda^T \delta x \Big|_0^\infty + \int_0^\infty (x^T C^T Q C + \lambda^T A + \dot{\lambda}^T)\delta x \, dt \\ &\quad + \operatorname{tr}\left\{\left[\int_0^\infty \left(x x^T C^T Q \frac{\partial C}{\partial p} + x \lambda^T \frac{\partial A}{\partial p}\right) dt - \nu \Delta p^T \Sigma^{-2}\right] d(\Delta p)\right\} - \mu x_0^T \, dx_0 \end{aligned}$$

Define $\lambda(t)$ to cause the coefficients of δx to vanish:

$$\dot{\lambda} = -A^T \lambda - C^T Q C x, \qquad \lambda(\infty) = 0$$

Let $\lambda = Sx$ with $S(\infty) = 0$ so that

$$\dot{\lambda} = \dot{S}x + S\dot{x}$$

Substituting for $\dot{\lambda}$ and $\dot{x}$ gives

$$\dot{S} = -A^T S - SA - C^T Q C, \qquad S(\infty) = 0$$

Hence, the steady-state solution for S is obtained from

$$0 = A^T S + SA + C^T Q C$$

With this definition of steady-state S, we have

$$d\bar{J} = \operatorname{tr}\left\{\left[\int_0^\infty x x^T \, dt \left(C^T Q \frac{\partial C}{\partial p} + S \frac{\partial A}{\partial p}\right) - \nu \Delta p^T \Sigma^{-2}\right] d(\Delta p)\right\} + x_0^T (S - \mu I) \, dx_0$$

Let

$$P = \int_0^\infty x x^T \, dt$$

It is straightforward to show that P is the solution to the steady-state Lyapunov equation:

$$0 = AP + PA^T + x_0 x_0^T$$

Thus, necessary conditions for a maximum with respect to Δp and x_0 are

$$\Delta p = \frac{1}{\nu}\Sigma^2 \left(\frac{\partial J}{\partial p}\right)^T$$

and

$$S x_0 = \mu x_0$$

where

$$\frac{\partial J}{\partial p_i} = \operatorname{tr}\left[P\left(C^T Q \frac{\partial C}{\partial p_i} + S \frac{\partial A}{\partial p_i}\right)\right]$$

It follows that $x_0 = \text{eigvec}(S)$ corresponding to the maximum eigenvalue of S, and

$$\Delta p = \sigma \Sigma \alpha$$

where α is a unit vector in the direction of

$$J_p = \Sigma \cdot \left(\frac{\partial J_w}{\partial p}\right)^T$$

Appendix B: Relationship to H_∞ Analysis and Design

H_∞ analysis is done in the frequency domain by considering the closed-loop transfer function matrix T from the disturbance vector w to the output vector z, where $z^T z$ is the integrand of the quadratic performance index J, i.e.,

$$z^T z \triangleq y^T Q y + u^T R u$$

where

$$z = \begin{bmatrix} \sqrt{Q}y \\ \sqrt{R}u \end{bmatrix}$$

and

$$z(s) = T(s) \cdot w(s)$$

Assuming a stable transfer function matrix T, performance criteria are defined as

$$\bar{J}_H(\omega) = \max_w \left\{ \frac{z^T(-j\omega)z(j\omega)}{w^T(-j\omega)w(j\omega)} \right\}$$

$$J_H = \max_\omega \left\{ \bar{J}_H(\omega) \right\}$$

It follows that

$$\bar{J}_H(\omega) = \max\left\{ \mathrm{eig}\left[T^T(-j\omega)T(j\omega) \right] \right\} = \left\{ \bar{\sigma}\left[T(j\omega) \right] \right\}^2$$

where $\bar{\sigma}(\ )$ = the maximum singular value of ( ).

If we make disturbances "cheap" by decreasing μ, we find that there is a minimum possible value of μ (call it $\mu_{\min}$).

For a fixed state feedback gain matrix K, $J_w \to \infty$ as $\mu \to \mu_{\min}$, because the w feedback destabilizes the system (one or more poles $\to$ the $j\omega$ axis).

For the minimax design (controls minimizing, disturbances maximizing), J_w may go indefinite or $\to \infty$ as $\mu \to \mu_{\min}$. In the latter case $\max|u|$ and $\max|w| \to \infty$; the system is *not* destabilized, but the bandwidth $\to \infty$. However, $\bar{\sigma}[T(j\omega)]$ has a low, constant value over the whole bandwidth. This is the H_∞ state feedback controller design.[4]

The easiest way to determine the H_∞ minimax design is to solve the Riccati equation (40) for decreasing values of μ until J_w either goes indefinite or $\to \infty$ (Ref. 2).

In examples we have calculated,[4] the infinite controller bandwidth is associated with one very fast mode that produces an impulsive change in the system state at time $t = 0^+$. Associated with this large bandwidth are infinite values of K and K_w. To keep the bandwidth reasonable, the H_∞ designer *must* use a compensator, since state feedback does not give reasonable results. Furthermore the designer must add fixed actuator prefilters or sensor postfilters (equivalent to specifying bounds on unmodeled high-frequency plant dynamics) to produce rolloff.

By use of the J_w design method the bandwidth is easily controlled by simply not decreasing μ all the way to $\mu_{\min}$; it is not necessary to force rolloff. The μ is determined, as in our helicopter example, to give a reasonable maximum value of the disturbance. If additional rolloff is required to avoid spillover of unmodeled higher-frequency dynamics, this can be done by adding actuator prefilters or sensor postfilters to the plant model.

Appendix C: Relationship to Real μ Analysis

If we were to use an infinity norm in defining $\sigma(\Delta p)$ instead of the two-norm as we did in Eq. (56), then[1]

$$\sigma_\infty \equiv \frac{1}{\mu_{\text{Doyle}}}$$

The q norm is defined as

$$\left\{ \sum_i \left(\frac{\Delta p_i}{\Sigma_{ii}} \right)^q \right\}^{1/q}$$

where $q = 2$ for the two norm used in this paper, and $q \to \infty$ for the ∞ norm (Doyle).

We can also define a performance parameter margin σ_ϵ (Ref. 11), where

$$\sigma_\epsilon = \min_{\Delta p} \sigma(\Delta p)$$

subject to the following constraint:

$$J_w(\Delta p) = (1+\epsilon)J_w(0)$$

and $\sigma(\Delta p)$ was defined in Eq. (56). This is the dual optimization problem to the maximum J_w with given σ problem defined earlier.

Approximations to σ_ϵ and σ_∞ are easily obtained using the performance sensitivity measure J_p defined in Eq. (59):

$$\sigma_\epsilon \cong \frac{1+\epsilon}{|J_p|}$$

$$\sigma_\infty \cong \frac{J_w}{|J_p|}$$

where J_p is evaluated at $p = p_{\text{nom}}$. These turned out to be very good approximations in the examples considered in the text.

Appendix D: Stability Parameter Margin by a Direct Method

For the helicopter example the transfer function from longitudinal cyclic stick deflection δ to forward position y is given by

$$\frac{y(s)}{\delta(s)} = \frac{X_\delta s^2 + (X_q M_\delta - X_\delta M_q)s - gM_\delta}{s\left[s^3 - (X_u + M_q)s^2 + (X_u M_q - X_q M_u)s + gM_u\right]} \triangleq \frac{n(s,p)}{d(s,p)}$$

where $p = [X_u, X_q, M_u, M_q, X_\delta, M_\delta]$ is the parameter vector, consisting of stability and control derivatives.

A fourth-order position-hold compensator is of the following form:

$$\frac{\delta(s)}{y(s)} = -\frac{n_1 s^3 + n_2 s^2 + n_3 s + n_4}{s^4 + d_1 s^3 + d_2 s^2 + d_3 s + d_4} \triangleq \frac{n_c(s)}{d_c(s)}$$

The closed-loop characteristic equation is

$$f(s) = d(s,p)d_c(s) + n(s,p)n_c(s) = 0$$

Let $s = j\omega$, and define

$$d(j\omega,p) \triangleq d_r + j\omega d_i$$

$$n(j\omega,p) \triangleq n_r + j\omega n_i$$

$$d_c(j\omega,p) \triangleq d_{cr} + j\omega d_{ci}$$

$$n_c(j\omega,p) \triangleq n_{cr} + j\omega n_{ci}$$

which implies

$$d_r = \omega^4 - (X_u M_q - X_q M_u)\omega^2$$

$$d_i = (X_u + M_q)\omega^2 + gM_u$$

$$n_r = -X_\delta \omega^2 - GM_\delta$$

$$n_i = X_q M_\delta - X_\delta M_q$$

$$d_{cr} = \omega^4 - d_2\omega^2 + d_4$$

$$d_{ci} = -d_1\omega^2 + d_3$$

$$n_{cr} = -n_2\omega^2 + d_4$$

$$n_{ci} = -n_1\omega^2 + n_3$$

Equating the real and imaginary parts of $f(j\omega)$ to zero then gives

$$f_r = d_r d_{cr} + n_r n_{cr} - \omega^2(d_i d_{ci} + n_i n_{ci}) = 0 \quad \text{(D1)}$$

$$f_i = d_i d_{cr} + d_r d_{ci} + n_i n_{cr} + n_r n_{ci} = 0 \quad \text{(D2)}$$

The stability parameter margin σ_∞ is defined as the minimum value of

$$\sigma = \left[(p-p_0)^T \Sigma^{-2}(p-p_0)\right]^{1/2}$$

over all ω and p, which yields a pole on the $j\omega$ axis; i.e., it satisfies Eqs. (D1) and (D2), where p_0 is the nominal value of p.

For the OH6A helicopter,

$$p_0^T = [-0.0257, 0.013, 1.26, -1.765, 0.086, -7.408]$$

$$\Sigma = \text{diag}[0.02, 0.01, 0.2, 0.2, 0.02, 0.5]$$

The LQG position-hold controller described in the main text has the following transfer function coefficients:

$$[n_1, n_2, n_3, n_4] = [9.3551, 20.3454, 6.4872, 2.7217]$$

$$[d_1, d_2, d_3, d_4] = [6.0108, 16.4906, 27.1643, 30.6097]$$

Changing the elements of p one at a time and holding the others at their nominal values, we found that the smallest scaled change in one parameter to produce instability was a 1.65σ increase in M_q.

For simultaneous changes in all the parameters, we used a parameter optimization code (PRMOPT) and found the stability parameter margin to be $\sigma_\infty = 1.467$, with

$$(p-p_0)^T \Sigma^{-1} = [0.158, 0.306, 0.336, 1.305, -0.174, -0.431]$$

Thus, it only takes a 1.30σ increase in M_q to produce instability when accompanied by the worst possible changes in the other parameters.

This computation confirms the computation in the main text, which used the performance parameter margin as the latter tends to infinity.

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